

ZYGMUND'S TYPE INEQUALITY TO THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract: In this paper we improve a result recently proved by Irshad et al. [On the Inequalities Concerning to the Polar Derivative of a Polynomial with Restricted Zeroes, Thai Journal of Mathematics, 2014 (Article in Press)] and also extend Zygmund's inequality to the polar derivative of a polynomial.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $P(z)$ be a polynomial of degree n , then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1.1)$$

inequality (1.1) is a well known result of S. Bernstein [1]. Equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin.

Inequality (1.1) was extended to L_p -norm $p \geq 1$ by Zygmund [2], who proved that if $P(z)$ is a polynomial of degree n , then

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |P'(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \quad (1.2)$$

Equality holds in (1.2) for $P(z) = \alpha z^n$, $|\alpha| \neq 0$. If we let $p \rightarrow \infty$ in (1.2), we get inequality (1.1).

Let α be a complex number. If $P(z)$ is a polynomial of degree n , then the polar derivative of $P(z)$ with respect to the point α , denoted by $D_\alpha P(z)$, is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

clearly $D_\alpha P(z)$ is a polynomial of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \rightarrow \infty} \left[\frac{D_\alpha P(z)}{\alpha} \right] = P'(z). \quad (1.3)$$

As an extension of (1.1) to the polar derivative, Aziz and Shah [3], have shown that if $P(z)$ is a polynomial of degree n , then for every real or complex number α with $|\alpha| > 1$ and for $|z| = 1$,

$$|D_\alpha P(z)| \leq n|\alpha| \max_{|z|=1} |P(z)| \quad (1.4)$$

As a generalization of (1.2) to the polar derivative Aziz et al. [4], proved the following result.

Theorem A. If $P(z)$ is a polynomial of degree n , then for every complex number α with $|\alpha| \geq 1$ and $p \geq 1$

$$\left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n(|\alpha| + 1) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \quad (1.5)$$

For the Class of polynomials having no zeros in $|z| < 1$, inequality (1.2) was improved by D-Bruijn [5] that if $P(z) \neq 0$ in $|z| < 1$, then for $p \geq 1$

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq nC_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \quad (1.6)$$

where

$$c_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\theta}|^p d\theta \right\}^{-\frac{1}{p}} \quad (1.7)$$

As an extension to the polar derivative. A. Aziz and N. Rather [6], proved the following generalization of (1.5). In fact they proved.

Theorem B. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$ and $p \geq 1$

$$\left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n(|\alpha| + 1)c_p \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \quad (1.8)$$

where C_p is defined by (1.7).

Recently, Irshad et al. [7] proved the following result.

Theorem C. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < K \leq 1$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq K$, $|\beta| \leq 1$ and $p \geq 1$

$$\begin{aligned} & \left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K)}{K + 1} \beta P(e^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}} \\ & \leq n \left(1 + |\alpha| + 2 \frac{(|\alpha| - K)}{K + 1} |\beta| \right) C_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \end{aligned} \quad (1.9)$$

where C_p is defined by (1.7).

In this paper we prove the following more general result which also generalize Theorem B and yields a number of known polynomial inequalities.

Theorem 1. If $P(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ be a polynomial of degree n which does not vanish in $|z| < K \leq 1$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq K$, $|\beta| \leq 1$ and $p \geq 1$

$$\begin{aligned} & \left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}} \\ & \leq n \left(1 + |\alpha| + 2 \frac{(|\alpha| - K^\mu)}{K^\mu + 1} |\beta| \right) C_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \end{aligned} \quad (1.10)$$

where C_p is defined by (1.7), or equivalently

$$\begin{aligned} & \left\| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right\|_p \\ & \leq n \left(1 + |\alpha| + 2 \frac{(|\alpha| - K^\mu)}{K^\mu + 1} |\beta| \right) \frac{\|P(e^{i\theta})\|_p}{\|1 + e^{i\phi}\|_p} \end{aligned} \quad (1.11)$$

Remark 1. If we choose $\mu = 1$ in (1.10), we get Theorem C and if we choose $\beta = 0$ and $K = 1$ in (1.10), we get Theorem B.

2. LEMMAS

For the proof of this theorem, we need the following lemmas. The first lemma is due to Gulshan Singh et al. [8].

Lemma 1. Let $P(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ be a polynomial of degree having all its zeros in the disk $|z| \leq K$, $K \leq 1$, then for every real or

complex number α with $|\alpha| \geq K$, $K \leq 1$ and for $|z| = 1$

$$|D_\alpha P(z)| \geq n \left(\frac{|\alpha| - K^\mu}{K^\mu} \right) |P(z)|$$

Lemma 2. Let $Q(z)$ be a polynomial of degree n having all its zeros in $|z| < K$, $K \leq 1$ and $P(z)$ is a polynomial of degree not exceeding that of $Q(z)$. If $|P(z)| \leq |Q(z)|$ for $|z| = K \leq 1$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq K$, $|\beta| \leq 1$

$$\left| zD_\alpha P(z) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(z) \right| \leq \left| zD_\alpha Q(z) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta Q(z) \right|$$

Proof. Since $|\lambda P(z)| \leq |P(z)| \leq |Q(z)|$, for $\lambda < 1$ and $|z| = K$, then for Rouché's Theorem $Q(z) - \lambda P(z)$ and $Q(z)$ have the same number of zeros in $|z| < K$. On the other hand by inequality $|P(z)| \leq |Q(z)|$ for $|z| = K$, any zero of $Q(z)$, that lies on $|z| = K$, is the zero of $P(z)$. Therefore, $Q(z) - \lambda P(z)$ has all its zero in the closed disk $|z| \leq K$. Hence by Lemma 1 for all real or complex numbers α with $|\alpha| \geq K$ and $|z| = 1$, we have

$$|zD_\alpha(Q(z) - \lambda P(z))| \geq n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} |Q(z) - \lambda P(z)| \quad (2.1)$$

Now consider a similar argument that for any value of β with $|\beta| < 1$, we have

$$\begin{aligned} |zD_\alpha(Q(z) - \lambda P(z))| &\geq n \left(\frac{|\alpha| - K^\mu}{K^\mu + 1} \right) |Q(z) - \lambda P(z)| \\ &> n|\beta| \left(\frac{|\alpha| - K^\mu}{K^\mu + 1} \right) |Q(z) - \lambda P(z)| \end{aligned} \quad (2.2)$$

where $|z| = 1$, resulting in

$$T(z) = |zD_\alpha Q(z) - \lambda zD_\alpha P(z)| + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} \{Q(z) - \lambda P(z)\} \neq 0 \quad (2.3)$$

where $|z| = 1$.

That is

$$T(z) = \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z) \right| - \lambda \left| zD_\alpha P(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z) \right| \neq 0 \quad (2.4)$$

for $|z| = 1$

We also conclude that

$$\left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z) \right| \geq \left| zD_\alpha P(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z) \right| \quad (2.5)$$

for $|z| = 1$.

If (2.5) is not true, then there is a point $z = z_0$ with $|z_0| = 1$, such that

$$\left| z_0 D_\alpha Q(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z_0) \right| < \left| z_0 D_\alpha P(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z_0) \right| \quad (2.6)$$

Take

$$\lambda = \frac{z_0 D_\alpha Q(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z_0)}{z_0 D_\alpha P(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z_0)} \quad (2.7)$$

then $|\lambda| < 1$ with this choice, we have from (2.4), $T(z_0) = 0$ for $|z_0| = 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| = 1$. For β with $|\beta| = 1$, (2.5) follows by continuity.

This completes the proof.

The next lemma is due to Aziz and Rather [4].

Lemma 3. If $P(z)$ is a polynomial of degree n such that $P(0) \neq 0$ and $Q(z) = z^n p \left(\frac{1}{\bar{z}} \right)$, then for every $p \geq 0$ and ϕ real

$$\int_0^{2\pi} \int_0^{2\pi} |Q'(e^{i\theta}) + e^{i\theta} P'(e^{i\theta})|^p d\theta d\phi \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta$$

3. PROOF OF THE THEOREM

Proof of the Theorem. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < K$, $K \leq 1$. By Lemma 2 for complex numbers α, β with $|\alpha| \geq K$, $|\beta| \leq 1$, we have

$$\left| z D_\alpha P(z) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(z) \right| \leq \left| z D_\alpha Q(z) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta Q(z) \right| \quad (3.1)$$

For every real ϕ and $\xi \geq 1$, we have

$$|\xi + e^{i\theta}| \geq |1 + e^{i\phi}|$$

which implies for any $p \geq 0$

$$\left\{ \int_0^{2\pi} |\xi + e^{i\phi}|^p d\phi \right\}^{\frac{1}{p}} \geq \left\{ \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \right\}^{\frac{1}{p}} \quad (3.2)$$

If $e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) \neq 0$, we can take

$$\xi = \frac{e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta Q(e^{i\theta})}{e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})}$$

where according to (3.1), $|\xi| \geq 1$. Now

$$\begin{aligned} & \int_0^{2\pi} \left| e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta Q(e^{i\theta}) \right. \\ & \quad \left. + e^{i\phi} \left[e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) \right] \right|^p d\phi \\ &= \left| e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) \right|^p \int_0^{2\pi} |\xi + e^{i\phi}|^p d\phi \\ &\geq \left| e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) \right|^p \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \end{aligned} \quad (3.3)$$

This inequality is trivially true if

$$e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) = 0$$

Integrating both sides of (3.3) with respect to θ from 0 to 2π , we have

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} \left| e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta Q(e^{i\theta}) \right. \\ & \quad \left. + e^{i\phi} \left[e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) \right] \right|^p d\theta d\phi \\ &\geq \int_0^{2\pi} \left| e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) \right|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \end{aligned} \quad (3.4)$$

Now for $0 \leq \theta < 2\pi$

$$\begin{aligned} & \left| e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta Q(e^{i\theta}) + e^{i\phi} \left[e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) \right] \right| \\ &= \left| \left[e^{i\theta} \{ nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta}) \} + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta Q(e^{i\theta}) \right] \right. \\ & \quad \left. + e^{i\phi} \left[e^{i\theta} \{ nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) \} + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) \right] \right| \end{aligned} \quad (3.5)$$

$$\begin{aligned}
&= \left| \left[e^{i\theta} \{nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta})\} + \alpha e^{i\theta} Q'(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta Q(e^{i\theta}) \right] \right. \\
&\quad \left. + e^{i\phi} \left[e^{i\theta} \{nP(e^{i\theta}) - e^{i\phi} P'(e^{i\theta})\} + \alpha e^{i\theta} P'(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right] \right| \\
&\hspace{25em} (3.6)
\end{aligned}$$

Since $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, we have $P(z) = z^n \overline{Q\left(\frac{1}{\bar{z}}\right)}$ and it can be easily verified that for $0 \leq \theta < 2\pi$

$$nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta}) = e^{i(n-1)\theta} \overline{Q'(e^{i\theta})}$$

and

$$nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta}) = e^{i(n-1)\theta} \overline{P'(e^{i\theta})}$$

From (3.6)

$$\begin{aligned}
&\left| e^{i\theta} D_\alpha Q(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta Q(e^{i\theta}) + e^{i\phi} \left[e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right] \right| \\
&= \left| \left[e^{i\theta} \{e^{i(n-1)\theta} \overline{P'(e^{i\theta})}\} \right] + \alpha e^{i\theta} [Q'(e^{i\theta}) + e^{i\phi} P'(e^{i\theta})] \right. \\
&\quad \left. + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta [Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta})] + e^{i\phi} e^{i\theta} e^{i(n-1)\theta} \overline{Q'(e^{i\theta})} \right| \\
&\hspace{25em} (3.7)
\end{aligned}$$

Therefore, (3.4) in conjunction with (3.7) gives

$$\begin{aligned}
&\left\{ \int_0^{2\pi} \int_0^{2\pi} |e^{i\theta} e^{i(n-1)\theta} \{\overline{P'(e^{i\theta})} + e^{i\phi} \overline{Q'(e^{i\theta})}\} + \alpha e^{i\phi} [Q'(e^{i\theta}) + e^{i\phi} P'(e^{i\theta})] \right. \\
&\quad \left. + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta [Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta})] \right|^p d\theta d\phi \Bigg\}^{\frac{1}{p}} \\
&\geq \left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \right\}^{\frac{1}{p}} \quad (3.8)
\end{aligned}$$

By Minkowski inequality, we have

$$\begin{aligned}
&\left\{ \int_0^{2\pi} |e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta})|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \right\}^{\frac{1}{p}} \\
&\geq \left\{ \int_0^{2\pi} \int_0^{2\pi} |Q'(e^{i\theta}) + e^{i\phi} P'(e^{i\theta})|^p d\theta d\phi \right\}^{\frac{1}{p}} \{1 + |\alpha|\} \\
&\quad + \left| n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta \right| \left\{ \int_0^{2\pi} \int_0^{2\pi} |Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta})|^p d\theta d\phi \right\}^{\frac{1}{p}}
\end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}
& \left\{ \int_0^{2\pi} |e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta})|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \right\}^{\frac{1}{p}} \\
& \leq \left\{ 2n^p \pi \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \{1 + |\alpha|\} \\
& \quad + 2n \left| \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \right| \left\{ 2\pi \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \\
& = \left[n(1 + |\alpha|) + 2n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} |\beta| \right] \left\{ 2\pi \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\end{aligned}$$

This implies

$$\begin{aligned}
& \left\{ \int_0^{2\pi} |e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \\
& \leq n \left(1 + |\alpha| + 2 \frac{(|\alpha| - K^\mu)}{K^\mu + 1} |\beta| \right) C_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\end{aligned}$$

where C_p is defined by (1.7),

or equivalently,

$$\begin{aligned}
& \left\| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right\| \\
& \leq n \left(1 + |\alpha| + 2 \frac{(|\alpha| - K^\mu)}{K^\mu + 1} |\beta| \right) \frac{\|P(e^{i\theta})\|_p}{\|1 + P(e^{i\phi})\|_p}
\end{aligned}$$

Hence the result.

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